Universality in invariant random-matrix models: Existence near the soft edge

E. Kanzieper and V. Freilikher
The Jack and Pearl Resnick Institute of Advanced Technology, Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel
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We consider two non-Gaussian ensembles of large Hermitian random matrices with strong level confinement and show that near the soft edge of the spectrum both scaled density of states and eigenvalue correlations follow so-called Airy laws inherent in the Gaussian unitary ensemble. This suggests that the invariant one-matrix models should display universal eigenvalue correlations in the soft-edge scaling limit. [S1063-651X(97)02803-1]

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I. INTRODUCTION

Unitary-invariant random-matrix models appear in many physical theories including nuclear physics, string theory, quantum chaos, and mesoscopic physics. They are completely defined by the joint distribution function

$$P[\mathbf{H}]=\frac{1}{\mathcal{Z}_N}\exp\{-\mathrm{Tr}V[\mathbf{H}]\} \tag{1}$$

of the entries of the $N\times N$ Hermitian matrix $\mathbf{H}$. In Eq. (1) the function $V[\mathbf{H}]$ is referred to as the "confinement potential" and it should provide the existence of the partition function $\mathcal{Z}_N$. A remarkable feature of this random matrix model is that, under certain conditions, a particular form of the confinement potential exerts no influence on the local eigenvalue correlations in the *bulk* scaling limit. More precisely, there is a class of strong, even confining potentials $V(\varepsilon)$, increasing at least as fast as $|\varepsilon|$ at infinity, for which the two-point kernel in the bulk of the eigenvalue spectrum follows the *sine* form in the large-N limit [1–4]:

$$K_{\mathrm{bulk}}(s,s')=\frac{\sin[\pi(s-s')]}{\pi(s-s')}. \tag{2}$$

This striking property, known as local universality, leads to the conclusion about universality of arbitrary n-point correlation functions $R_n(s_1,\ldots,s_n)=\det[K(s_i,s_j)]_{i,j=1,\ldots,n}$ $(n>1)$ in the local regime. In contrast, the global characteristics of the eigenspectrum, such as the density of states or one-point Green's function, display a great sensitivity to the details of confinement potential [3].

Less is known about eigenvalue correlations near the *soft edge*, which is of special interest in the matrix models of two-dimensional quantum gravity [5]. In the early study [6] the behavior of the density of states near the tail of eigenvalue support has been explored. The authors of Ref. [6] showed that there is a universal crossover from a nonzero density of states to a vanishing density of states that is independent of the confining potential in the soft-edge scaling limit. Whereas the universal behavior of the density of states in the soft-edge scaling limit has been proven, the (supposed) universality of n-point correlations has not yet been considered.

The problem we address in this work is whether the eigenvalue correlations in ensembles of large random matrices also possess a universal behavior in the soft-edge scaling limit. To provide an answer to this question, we first quote some results for the Gaussian unitary ensemble (GUE), which has received the most study near the soft edge, and then we turn to the consideration of eigenspectra of two strongly non-Gaussian ensembles of random matrices associated with quartic and sextic confining potentials.

In the soft-edge scaling limit, the GUE is characterized by the *Airy* two-point kernel [7]

$$K_{\mathrm{GUE}}(s,s')=\frac{\mathrm{Ai}(s)\mathrm{Ai}'(s')-\mathrm{Ai}(s')\mathrm{Ai}'(s)}{s-s'}, \tag{3}$$

whose spectral properties were studied in detail in Ref. [8]. As a consequence of Eq. (3), the scaled density of states, unlike in the case of bulk scaling limit, cannot be taken as being approximately constant and changes in accordance with the Airy law

$$\nu_{\mathrm{GUE}}(s)=\left(\frac{d}{ds}\mathrm{Ai}(s)\right)^2-s[\mathrm{Ai}(s)]^2 \tag{4a}$$

with asymptotes

$$\nu_{\mathrm{GUE}}(s)=\begin{cases}\dfrac{\sqrt{|s|}}{\pi}-\dfrac{\cos(4|s|^{3/2}/3)}{4\pi|s|}, & s\to-\infty\\[2ex] \dfrac{1}{8\pi s}\exp(4s^{3/2}/3), & s\to+\infty.\end{cases} \tag{4b}$$

Our following treatment of non-Gaussian random matrix ensembles with strong level confinement will be built upon the orthogonal polynomial technique [9], allowing us to express the two-point kernel for the random-matrix ensemble defined by Eq. (1) through the polynomials $P_N(\varepsilon)$ orthogonal on the whole real axis with respect to the weight $\exp\{-2V(\varepsilon)\}$. We fix the polynomials P_n satisfying the three-term recurrence formula

$$\varepsilon P_n=a_{n+1}P_{n+1}+a_nP_{n-1} \tag{5}$$

to be orthonormal,

$$\int_{-\infty}^{+\infty}d\varepsilon P_n(\varepsilon)P_m(\varepsilon)\exp\{-2V(\varepsilon)\}=\delta_{nm}. \tag{6}$$

Under these conditions the two-point kernel reads

$$K_N(\varepsilon,\varepsilon')=a_N\frac{\psi_N(\varepsilon')\psi_{N-1}(\varepsilon)-\psi_N(\varepsilon)\psi_{N-1}(\varepsilon')}{\varepsilon'-\varepsilon},\quad (7)$$

where [10] $a_N=k_{N-1}/k_N$ [k_N is a leading coefficient of the orthogonal polynomial $P_N(\varepsilon)$] and the wave functions $\psi_N(\varepsilon)=P_N(\varepsilon)\exp\{-V(\varepsilon)\}$ have been introduced. Inasmuch as our concern is with the matrices of large dimensions $N\gg1$, only the asymptotes of the wave functions ψ_N are needed and also a meaningful *scaling limit* should be constructed. Quite generally, this can be done by passing from the initial energy variable ε to a new scaled variable s that remains finite as $N\to\infty$: $\varepsilon=\varepsilon(N,s)=\varepsilon_s$. Then the scaled two-point kernel is determined by the formula

$$K(s,s')=\lim_{N\to\infty}K_N(\varepsilon_s,\varepsilon_{s'})\frac{d\varepsilon_s}{ds}.\quad (8)$$

II. QUARTIC CONFINEMENT POTENTIAL

We choose the quartic confinement potential in the form $V(\varepsilon)=\frac{1}{2}\varepsilon^4$. In this case the differential equation for $\psi_n(\varepsilon)$ [the index n is an arbitrary positive integer] can be obtained by Shohat's method [11,12]:

$$\frac{d^2}{d\varepsilon^2}\psi_n(\varepsilon)-\left[\frac{d}{d\varepsilon}\ln\varphi_n(\varepsilon)\right]\frac{d}{d\varepsilon}\psi_n(\varepsilon)+Q_n(\varepsilon)\psi_n(\varepsilon)=0,\quad (9a)$$

$$\varphi_n(\varepsilon)=a_{n+1}^2+a_n^2+\varepsilon^2,\quad (9b)$$

$$Q_n(\varepsilon)=\left(6\varepsilon^2-4\varepsilon^6-\frac{4\varepsilon^4}{\varphi_n(\varepsilon)}\right)+4a_n^2\left(4\varphi_n(\varepsilon)\varphi_{n-1}(\varepsilon)+1-4a_n^2\varepsilon^2-4\varepsilon^4-\frac{2\varepsilon^2}{\varphi_n(\varepsilon)}\right).\quad (9c)$$

Here a_n is the recursion coefficient entering the corresponding three-term recurrence formula for the given set of orthogonal polynomials. Also, the following exact relation takes place:

$$\psi_{n-1}(\varepsilon)=\frac{\psi_n'(\varepsilon)+\psi_n(\varepsilon)[V'(\varepsilon)+4\varepsilon a_n^2]}{4a_n\varphi_n(\varepsilon)}.\quad (10)$$

Hereafter we shall be interested in the behavior of the wave function ψ_n near the soft band edge D_n in the limit $n=N\gg1$. In this case the end point of the spectrum $D_N=2a_N$, where [13]

$$a_N=\left(\frac{N}{12}\right)^{1/4}[1+O(N^{-2})]\quad (11)$$

and

$$\varphi_N(\varepsilon)=2a_N^2+\varepsilon^2+O(N^{-1/2}).\quad (12)$$

Let us move the spectrum origin to its end point D_N, making the replacement $\varepsilon=D_N+t$, and denote $\hat\psi_N(t)=\psi_N(\varepsilon-D_N)$. It is straightforward to show that this function obeys the equation

$$\frac{d^2}{dt^2}\hat\psi_N(t)-18D_N^5t\hat\psi_N(t)=0\quad (13)$$

in the asymptotic limit $N\gg1$. When deriving this equation we supposed the characteristic energy scale $t_v(N)=|d\ln\hat\psi_N(t)/dt|^{-1}$ of the variation of $\hat\psi_N(t)$ to be much smaller than the band edge D_N.

The solution to Eq. (13) can be written through the Airy function $y(x)=\mathrm{Ai}(x)$ satisfying the differential equation $y''(x)-xy(x)=0$:

$$\hat\psi_N(t)=\lambda_N\mathrm{Ai}[t(18D_N^5)^{1/3}].\quad (14)$$

One can check that the condition $t_v(N)\sim O(N^{-5/12})\ll D_N$ is fulfilled. The coefficient λ_N entering Eq. (14) still remains unknown.

To compute the two-point kernel Eq. (7), we have to correctly determine the asymptotic behavior of $\hat\psi_{N-1}(t)$. This can be done by means of the asymptotic analysis of the exact relation Eq. (10), which in the large-N limit comes down to

$$\hat\psi_{N-1}(t)=\hat\psi_N(t)+\frac{1}{3D_N^3}\frac{d}{dt}\hat\psi_N(t).\quad (15)$$

It is convenient to define the soft-edge scaling limit as

$$\varepsilon_s=D_N+\frac{s}{(18D_N^5)^{1/3}}.\quad (16)$$

Then the two-point kernel Eq. (7) and the density of states $K(\varepsilon_s,\varepsilon_s)$ are given by the formulas

$$K(\varepsilon_s,\varepsilon_{s'})=\lambda_N^2\left(\frac{3}{2}D_N^4\right)^{1/3}K_{\mathrm{GUE}}(s,s')\quad (17a)$$

and

$$\nu(\varepsilon_s)=\lambda_N^2\left(\frac{3}{2}D_N^4\right)^{1/3}\nu_{\mathrm{GUE}}(s),\quad (17b)$$

respectively. The latter expression provides a possibility to determine the unknown constant λ_N by fitting the soft-edge density of states Eq. (17b) to the bulk density of states [4]

$$\nu_{\mathrm{bulk}}(\varepsilon_s)=\frac{D_N^3}{\pi}\sqrt{1-\left(\frac{\varepsilon_s}{D_N}\right)^2}\left[1+2\left(\frac{\varepsilon_s}{D_N}\right)^2\right]\quad (18)$$

taken near the end point of the spectrum Eq. (16) provided $1\ll s\ll D_N^{5/3}$. Equations (18), (17b), (16), and (4b) yield the value $\lambda_N^2=(12D_N)^{1/3}$. Now, making use of Eqs. (17a), (16), and (8), we arrive at the following expressions for the two-point kernel in the soft-edge scaling limit:

$$K_{\mathrm{soft}}(s,s')=K_{\mathrm{GUE}}(s,s').\quad (19)$$

Thus we conclude that the two-point kernel and the density of states, computed for the random-matrix ensemble with the quartic confining potential in the soft-edge scaling limit, coincide exactly with those for the GUE.

III. SEXTIC CONFINEMENT POTENTIAL

Now we turn to another ensemble of random matrices that is characterized by the confinement potential $V(\varepsilon)=\frac{1}{12}\varepsilon^6$. Corresponding wave functions ψ_n satisfy the same differential equation (9a), but with [14]

$$Q_n(\varepsilon)=-\frac{1}{4}\varepsilon^{10}+\frac{5}{2}\varepsilon^4-\frac{1}{2}\varepsilon^5\left[\frac{d}{d\varepsilon}\ln\varphi_n(\varepsilon)\right] +a_n^2\varphi_n(\varepsilon)\varphi_{n-1}(\varepsilon)-\left(\varepsilon^5+\pi_n(\varepsilon)-\frac{d}{d\varepsilon}\right)\pi_n(\varepsilon) -2\varepsilon\frac{\pi_n(\varepsilon)}{\varphi_n(\varepsilon)}(2\varepsilon^2+a_n^2+a_{n+1}^2), \tag{20a}$$

$$\pi_n(\varepsilon)=a_n^2\varepsilon(a_{n-1}^2+a_n^2+a_{n+1}^2+\varepsilon^2), \tag{20b}$$

and

$$\varphi_n(\varepsilon)=a_{n+1}^2(a_{n+2}^2+a_{n+1}^2+a_n^2)+a_n^2(a_{n+1}^2+a_n^2+a_{n-1}^2) +\varepsilon^2(a_{n+1}^2+a_n^2+\varepsilon^2). \tag{20c}$$

Also, the following relationship holds for arbitrary n:

$$\psi_{n-1}(\varepsilon)=\frac{\psi_n'(\varepsilon)+\psi_n(\varepsilon)[V'(\varepsilon)+\pi_n(\varepsilon)]}{a_n\varphi_n(\varepsilon)}. \tag{21}$$

The asymptotic analysis of the solution to the second-order differential equation near the end point of the spectrum is quite similar to that in the preceding section. Therefore, we sketch only its main points.

For $n=N\gg1$ the recursion coefficient [15]

$$a_N=\left(\frac{N}{10}\right)^{1/6}[1+O(N^{-2})], \tag{22}$$

and

$$\varphi_N(\varepsilon)=6a_N^4+\varepsilon^2(\varepsilon^2+2a_N^2)+O(N^{-1/2}), \tag{23}$$

$$\pi_N(\varepsilon)=a_N^2\varepsilon(\varepsilon^2+3a_N^2)+O(N^{-1/3}). \tag{24}$$

Introducing the shifted energy variable $\varepsilon=D_N+t$, we are able to rewrite the differential equation (9a) for the function $\hat{\psi}_N(t)=\psi_N(\varepsilon-D_N)$ in the form

$$\frac{d^2}{dt^2}\hat{\psi}_N(t)-\frac{225}{128}D_N^9t\hat{\psi}_N(t)=0, \tag{25}$$

assuming that the characteristic energy scale $t_v(N)=|d\ln\hat{\psi}_N(t)/dt|^{-1}$ of the variation of $\hat{\psi}_N(t)=\psi_N(\varepsilon-D_N)$ is much smaller than the band edge D_N.

The solution of Eq. (25) takes the form

$$\hat{\psi}_N(t)=\lambda_N\mathrm{Ai}\left[\left(\frac{225D_n^9}{128}\right)^{1/3}t\right], \tag{26}$$

with the coefficient λ_N that will be determined later by the same fitting arguments. The assumption $t_v(N)\ll D_N$ is obviously fulfilled.

To get the asymptotic behavior of $\hat{\psi}_{N-1}(t)$ in the large-N limit, we simplify Eq. (21) to

$$\hat{\psi}_{N-1}(t)=\hat{\psi}_N(t)+\frac{16}{15D_N^5}\frac{d}{dt}\hat{\psi}_N(t). \tag{27}$$

It is convenient to define the soft-edge scaling limit as

$$\varepsilon_s=D_N+\frac{s}{D_N^3}\left(\frac{128}{225}\right)^{1/3}. \tag{28}$$

Then the two-point kernel is

$$K(\varepsilon_s,\varepsilon_{s'})=\lambda_N^2D_N^2\left(\frac{15}{32}\right)^{1/3}K_{\mathrm{GUE}}(s,s'), \tag{29a}$$

while the density of states takes the form

$$\nu(\varepsilon_s)=\lambda_N^2D_N^2\left(\frac{15}{32}\right)^{1/3}\nu_{\mathrm{GUE}}(s). \tag{29b}$$

The fitting arguments, based on the expansion of Eq. (29b) and of the bulk density of states [4]

$$\nu_{\mathrm{bulk}}(\varepsilon_s)=\frac{D_N^5}{16\pi}\sqrt{1-\left(\frac{\varepsilon_s}{D_N}\right)^2}\left[3+4\left(\frac{\varepsilon_s}{D_N}\right)^2+8\left(\frac{\varepsilon_s}{D_N}\right)^4\right] \tag{30}$$

near the soft edge (when $1\ll s\ll D_N^3$), yield $\lambda_N^2=D_N(15/4)^{1/3}$. Combining Eqs. (29a), (28), and (8), we end with the following expression for the two-point kernel in the soft-edge scaling limit:

$$K_{\mathrm{soft}}(s,s')=K_{\mathrm{GUE}}(s,s'). \tag{31}$$

This formula demonstrates that in the soft-edge scaling limit the eigenlevel properties for the random-matrix ensemble with sextic confinement potential are determined by the same Airy law that is inherent in GUE.

IV. CONCLUDING REMARKS

We have considered the correlations of the eigenlevels near the soft edge for two strongly non-Gaussian ensembles of large random matrices possessing unitary symmetry and associated with quartic and sextic confinement potentials. Our treatment has been based on the analysis of the second-order differential equations for the corresponding wave functions near the soft edge. In both cases it was found that correlations between appropriately scaled eigenvalues are universal and characterized by the Airy two-point kernel Eq. (3), which previously has been found for GUE.

Together with the universal behavior of the density of states, previously proven in Ref. [6], the consideration presented gives a strong impression that spectral correlations in invariant ensembles of large random matrices with a rather strong and monotonic confinement potential are indeed universal near the soft edge.

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